

Energy and angular momentum of general 4-dimensional stationary axi-symmetric spacetime in teleparallel geometry

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We derive an exact general axi-symmetric solution of the coupled gravitational and electromagnetic fields in the tetrad theory of gravitation. The solution is characterized by four parameters M (mass), Q (charge), a (rotation) and L (NUT). We then, calculate the total exterior energy using the energy-momentum complex given by Møller in the framework of Weitzenböck geometry. We show that the energy contained in a sphere is shared by its interior as well as exterior. We also calculate the components of the spatial momentum to evaluate the angular momentum distribution. We show that the only non-vanishing components of the angular momentum is in the Z direction.

1. Introduction

A general stationary axi-symmetric object, with gravitomagnetic monopole and dipole moments associated with nonzero values of the NUT and Kerr parameters L and a respectively, is described by Kerr-NUT (Newman-Unti-Tamburino) spacetime [1] which is a useful model for exploring gravitomagnetism [2]. The Kerr-NUT spacetime and its spacial cases are all belong to larger class of stationary axi-symmetric type D. Carter [3] has found vacuum solutions of Einstein equations for which the Hamilton-Jacobi equation of the geodesic is separable. The stability of the general axi-symmetric spacetime is probed by studying their perturbation by fields of various spin [4, 5].

Einstein's general relativity (GR) is a very successful theory in describing long distance phenomena. However, it encounters from serious difficulties on short distances. The main problem appears in all attempts is the quantization [6, 7]. Also, the Lagrangian structure of GR differs from the ordinary microscopic gauge theories. In particular, a covariant conserved energy-momentum tensor for the gravitational field can not be constructed in the framework of GR. Consequently, the study of alternative models of gravity is justified from the physical as well as from the mathematical point of view. Even in the case when GR is a unique true theory of gravity, consideration of close alternative models can shed light on the properties of GR itself.

It is well known that in GR an energy-momentum tensor fails to satisfy some certain conditions [8]. This is usually related to the equivalence principle which implies that the

gravitational field can not be detected at a point as a covariant object. This can be viewed as a purely differential-geometric fact. Since the components of the metric tensor are managed by a system of second order partial differential equations therefore, the energy momentum quantity has to be a local tensor constructed out from the metric components and their first order derivatives. The corresponding theorem of (pseudo) Riemannian geometry states that every expression of such type is trivial. It is natural to expect that this objection for the existence of a gravitational energy-momentum tensor is directly related to the geometric properties of the (pseudo) Riemannian manifold. This objection can be lifted in alternative model, even connected with the geometry of the manifold.

In recent time teleparallel structures on spacetime have evoked a considerable interest for various reasons. They were considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory ([9] ~ [13] and references therein) or metric-affine gravity [14]. Physics relevant to geometry may be related to the teleparallel description of gravity [15, 16]. Teleparallel approach is used for positive-gravitational-energy proof [17]. The relation between spinor Lagrangian and teleparallel theory is established [18].

Møller has shown that the problem of the energy-momentum complex has no solution in the framework of gravitational field theories based on Riemannian spacetime [19]. In a series of papers, [19, 20, 21] he was able to obtain a general expression for a satisfactory energy-momentum complex in the absolute parallelism space. The Lagrangian formulation of the theory was given by Pellegrini and Plebanski [22]. Quite independently Hayashi and Nakano [23] formulated the tetrad theory of gravitation as a gauge theory of the spacetime translation group. In these attempts, the admissible Lagrangians were limited by the assumption that the field equations has the Schwarzschild solution. Møller [24] abandoning this assumption and look for a wider class of Lagrangians by constructing a new field theory. His aim was to get a theory free from singularity while retaining the main merits of GR as far as possible. Meyer [25] showed that Møller's theory is a special case of the Poincaré gauge theory [26, 27]. Sáez [28] generalized Møller theory into a scalar tetradic theory of gravitation.

The tetrad theory of gravitation based on the geometry of absolute parallelism [22]~[34] can be considered as the closest alternative to GR and it has a number of attractive features both from the geometrical and physical viewpoints. Absolute parallelism is naturally formulated by gauging spacetime translations. Translations are closely related to the group of general coordinate transformations which underlies GR. Therefore, the energy-momentum tensor represents the matter source in the field equation for the gravitational field is just like in GR.

The tetrad formulation of gravitation was considered by Møller in connection with attempts to define the energy of gravitational field [24, 35]. For a satisfactory description of the total energy of an isolated system it is necessary that the energy-density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [36, 37].

The main aim of the present paper is to derive a general axi-symmetric solution in

the tetrad theory of gravitation for the coupled gravitational and electromagnetic fields. Using this solution we then, calculate the energy and angular momentum using the *energy-momentum complex* given by Møller [24] and Mikhail *et.al* [38]. In §2 we derive the field equations of the coupled gravitational and electromagnetic fields. In §3 we obtain a new exact analytic axi-symmetric solution characterized by four parameters in the tetrad theory of gravitation. In §4 we calculate the energy and angular momentum distribution of this solution using Møller's energy-momentum complex [24]. The final section is devoted to discussion and conclusion.

2. Tetrad theory of gravitation and electromagnetism

In the Weitzenböck geometry the fundamental field variables describing gravity are a quadruplet of parallel vector fields [26] $e_i^{\mu*}$, which we call the tetrad field in this paper, characterized by

$$D_\nu e_i^\mu = \partial_\nu e_i^\mu + \Gamma^\mu_{\lambda\nu} e_i^\lambda = 0, \quad (1)$$

where $\Gamma^\mu_{\lambda\nu}$ defines the nonsymmetric affine connection coefficients. The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu,$$

with the Minkowski metric $\eta_{ij} = \text{diag}(+1, -1, -1, -1)^\dagger$. The curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$, given by Eq. (1) i.e. $R^\sigma_{\mu\nu\lambda}(\Gamma)$, is identically vanishing [26].

The gravitational Lagrangian L_G is an invariant constructed from $g_{\mu\nu}$ and the contorsion tensor $\gamma_{\mu\nu\rho}$ given by

$$\gamma_{\mu\nu\rho} = e^i_\mu e_{i\nu;\rho}, \quad (2)$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general gravitational Lagrangian density invariant under parity operation is given by the form [23, 26, 38]

$$\mathcal{L}(e)_G = bL_G(e) = e (\alpha_1 \Phi^\mu \Phi_\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu}) \quad (3)$$

and Φ_μ being the basic vector field defined by

$$\Phi_\mu = \gamma^\rho_{\mu\rho}, \quad (4)$$

where $e = \det(e^a_\mu)$. Here α_1, α_2 , and α_3 are constants determined such that the theory coincides with general relativity in the weak fields [23, 24]:

$$\alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \frac{\lambda}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa}(1 - \lambda), \quad (5)$$

*Latin indices (i, j, k, \dots) designate the vector number, which runs from (0) to (3), while Greek indices (μ, ν, ρ, \dots) designate the world-vector components running from 0 to 3. The spatial part of Latin indices is denoted by (a, b, c, \dots) , while that of Greek indices by $(\alpha, \beta, \gamma, \dots)$.

[†]Latin indices are raising and lowering with the aid of η_{ij} and η^{ij} .

where κ is the Einstein constant and λ is a free dimensionless parameter[‡]. The vanishing of this parameter makes the theory coincides with GR formulated in teleparallel geometry.

The electromagnetic Lagrangian density $L_{e.m.}$ is [32]

$$L_{e.m.} = -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, \text{ with } F_{\mu\nu} \text{ being given by } {}^{\S}F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \quad (6)$$

where A_{μ} is the vector potential. The variation of the gravitational Lagrangian given by Eq. (3) with respect to the tetrad field gives one equation of motion. Møller assumed that the energy-momentum tensor of matter fields, i.e., $T_{\mu\nu}$ is symmetric. Therefore Møller divided the resulting field equation into symmetric part and skew symmetric part [24].

The gravitational and electromagnetic field equations for the system described by $L_G + L_{e.m.}$ are the following:

$$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (7)$$

$$K_{\mu\nu} = 0, \quad (8)$$

$$\partial_{\nu}(\sqrt{-g}F^{\mu\nu}) = 0 \quad (9)$$

with $G_{\mu\nu}$ being the Einstein tensor of GR. Here $H_{\mu\nu}$ and $K_{\mu\nu}$ are defined by

$$H_{\mu\nu} \stackrel{\text{def.}}{=} \lambda \left[\gamma_{\rho\sigma\mu}\gamma^{\rho\sigma}{}_{\nu} + \gamma_{\rho\sigma\mu}\gamma_{\nu}{}^{\rho\sigma} + \gamma_{\rho\sigma\nu}\gamma_{\mu}{}^{\rho\sigma} + g_{\mu\nu} \left(\gamma_{\rho\sigma\lambda}\gamma^{\lambda\sigma\rho} - \frac{1}{2}\gamma_{\rho\sigma\lambda}\gamma^{\rho\sigma\lambda} \right) \right], \quad (10)$$

and

$$K_{\mu\nu} \stackrel{\text{def.}}{=} \lambda \left[\Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_{\rho} \left(\gamma^{\rho}{}_{\mu\nu} - \gamma^{\rho}{}_{\nu\mu} \right) + \gamma_{\mu\nu}{}^{\rho}{}_{;\rho} \right], \quad (11)$$

and they are symmetric and antisymmetric tensors, respectively. The energy-momentum tensor $T^{\mu\nu}$ is given by

$$T_{\mu\nu} = -g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \quad (12)$$

It can be shown [26] that the tensors, $H_{\mu\nu}$ and $K_{\mu\nu}$, consist of only those terms which are linear or quadratic in the axial-vector part of the torsion tensor, a_{μ} , defined by

$$a_{\mu} \stackrel{\text{def.}}{=} \frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}, \quad (13)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is defined by

$$\epsilon_{\mu\nu\rho\sigma} \stackrel{\text{def.}}{=} \sqrt{-g}\delta_{\mu\nu\rho\sigma}, \quad (14)$$

with $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$. Therefore, both $H_{\mu\nu}$ and $F_{\mu\nu}$ vanish if the a_{μ} is vanishing. In other words, when the a_{μ} is found to vanish from the antisymmetric part of the field equations, (8), the symmetric part (7) coincides with the Einstein equation formulated in the Weitzenböck spacetime.

[‡]Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

[§]Heaviside-Lorentz rationalized units will be used.

3. Exact Analytic Solutions

In this section we will seek for a solution satisfying the following conditions: The parallel vector fields having the form

$$e_\mu{}^k = \delta_\mu{}^k + M l_\mu l^k + \frac{Q^2 - L^2}{2} m_\mu m^k. \quad (15)$$

Here M , Q^2 and L^2 are free parameters, l_μ and m_μ are quantities satisfying the conditions

$$\eta^{\mu\nu} l_\mu l_\nu = 0, \quad \eta^{\mu\nu} m_\mu m_\nu = 0, \quad \eta^{\mu\nu} l_\mu m_\nu = 0, \quad (16)$$

and l^k and m^k are defined by

$$l^k \stackrel{\text{def.}}{=} \delta^k{}_\mu \eta^{\mu\nu} l_\nu, \quad m^k \stackrel{\text{def.}}{=} \delta^k{}_\mu \eta^{\mu\nu} m_\nu. \quad (17)$$

Applying (15) to the field equations (7)~(9) we obtain the values of l_μ and m_μ in the form

$$l_0 = \sqrt{\Xi}, \quad l_\alpha = \frac{2\sqrt{\Xi}}{\Upsilon + r^2 + h^2} \left[\Omega x_\alpha + \frac{a^2 x_3 \delta^3{}_\alpha}{\Omega} - \epsilon_{\alpha\beta 3} a x^\beta \right], \quad m_\mu = \frac{l_\mu}{\sqrt{\Omega}}, \quad (18)$$

where

$$\Xi = \frac{\Omega}{\Upsilon}, \quad \Upsilon = \sqrt{(r^2 - a^2)^2 + 4a^2 z^2}, \quad \Omega = \sqrt{\frac{r^2 - a^2 + \Upsilon}{2}}, \quad (19)$$

with $r = \sqrt{x^2 + y^2 + z^2}$ and $\epsilon_{\alpha\beta\gamma}$ are the three dimensional totally antisymmetric tensor with $\epsilon_{123} = 1$ and a is the angular momentum per unit mass. The form of the electromagnetic potential A_μ has the expression

$$A_\mu = \frac{-q}{4\pi} \sqrt{\Xi} l_\mu, \quad \text{and the Maxwell field } F_{\mu\nu} = \frac{q}{4\pi} \left[\left(\sqrt{\Xi} l_\nu \right)_{,\mu} - \left(\sqrt{\Xi} l_\mu \right)_{,\nu} \right], \quad (20)$$

where Ξ and l_μ are defined in Eqs. (18) and (19).

For the solution given by Eqs. (18) and (20), the axial vector part a_μ of the torsion tensor vanishes,

$$a_\mu = 0, \quad (21)$$

and the metric is identical to Kerr-NUT metric in GR.

Writing explicitly the tetrad (15) using (18) and (19) we obtain*

$$e^{(0)}_0 = 1 - \frac{(2M\rho - Q^2 + L^2)\rho^2}{2\rho_1},$$

*We will denote the symmetric part by $(\)$, for example, $A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the square bracket $[\]$, $A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu})$.

$$\begin{aligned}
e^{(0)}_{\alpha} &= \left\{ -\left(n_{\alpha} - \frac{a}{\rho} \epsilon_{\alpha j 3} n^j\right) - \frac{a^2 z}{\rho^2} \delta_{\alpha}^3 \right\} \frac{(2M\rho - Q^2 + L^2)\rho^4}{2\rho_1(\rho^2 + a^2)} = -e^{(l)}_0, \\
e^{(l)}_{\beta} &= \delta^l_{\beta} + \left\{ x^l x_{\beta} - 2\frac{a}{\rho} \epsilon_{k3(\beta} x^l) x^k + \frac{a^2}{\rho^2} \left[\epsilon_k^{l3} \epsilon_{m\beta 3} x^k x^m + z \left(\{ \rho x^l - \frac{a}{\rho} \epsilon_{k3}^l x^k \} \delta_{\beta}^3 \right. \right. \right. \\
&\quad \left. \left. \left. + \{ \rho x_{\beta} - \frac{a}{\rho} \epsilon_{k3\beta} x^k \} \delta^{l3} \right) \right] + \frac{a^4}{\rho^4} z^2 \delta_{\beta 3} \delta^{3l} \right\} \frac{(2M\rho - Q^2 + L^2)\rho^4}{2\rho_1(\rho^2 + a^2)^2}, \tag{22}
\end{aligned}$$

with δ^l_{β} being the Kronecker delta [32] and

$$\rho_1 = \rho^4 + a^2 z^2.$$

Solution (22) with Eq. (20) satisfy the field equations (7)~(9) and the associated metric has the following form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{(2Mr_1 - a^2 - Q^2 + L^2)r_1^2}{r_1^4 + a^2 z^2} \left\{ dt + \frac{zdz}{r_1} + \frac{r_1(xdx + ydy) + a(xdy - ydx)}{r_1^2 + a^2} \right\}^2, \tag{23}$$

with r_1 is the radial parameter related to the Cartesian radius r by

$$r_1^4 - (r^2 - a^2)r_1^2 - a^2 z^2 = 0.$$

4. The energy and angular momentum associated with the general axi-symmetric solution

The superpotential is given by [24, 38]

$$\mathcal{U}_{\mu}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi\rho\sigma}^{\tau\nu\lambda} [\Phi^{\rho} g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^{\chi\rho\sigma} - (1 - 2\lambda) g_{\tau\mu} \gamma^{\sigma\rho\chi}], \tag{24}$$

where $P_{\chi\rho\sigma}^{\tau\nu\lambda}$ is

$$P_{\chi\rho\sigma}^{\tau\nu\lambda} \stackrel{\text{def.}}{=} \delta_{\chi}^{\tau} g_{\rho\sigma}^{\nu\lambda} + \delta_{\rho}^{\tau} g_{\sigma\chi}^{\nu\lambda} - \delta_{\sigma}^{\tau} g_{\chi\rho}^{\nu\lambda} \tag{25}$$

with $g_{\rho\sigma}^{\nu\lambda}$ being a tensor defined by

$$g_{\rho\sigma}^{\nu\lambda} \stackrel{\text{def.}}{=} \delta_{\rho}^{\nu} \delta_{\sigma}^{\lambda} - \delta_{\sigma}^{\nu} \delta_{\rho}^{\lambda}. \tag{26}$$

The energy-momentum density is defined by [24]

$$\tau_{\mu}^{\nu} = \mathcal{U}_{\mu}^{\nu\lambda},_{\lambda}. \tag{27}$$

The energy E contained in a sphere with radius R is expressed by the volume integral [39]

$$E(R) = \int_{r=R} \int \int \tau_0^0 d^3x. \quad (28)$$

For convenience of the calculations, we work for small values of the rotation parameter a , so we will neglect the terms beyond its fourth order. With this approximation, we have the covariant components of the parallel vector fields (22) as

$$\begin{aligned} e^{(0)}_0 &= 1 - \frac{1}{r} \left(M - \frac{Q^2 - L^2}{2r} \right) - \frac{a^2}{2r^3} \left[\left(M - \frac{Q^2 - L^2}{r} \right) - \left(3M - \frac{2(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right] \\ &\quad - \frac{a^4}{8r^5} \left[\left(3M - \frac{4(Q^2 - L^2)}{r} \right) - \left(30M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} + \left(35M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right], \\ e^{(0)}_\alpha &= - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{n_\alpha}{r} + \frac{a}{r^2} \left(M - \frac{Q^2 - L^2}{2r} \right) \epsilon_{\alpha\beta 3} n^\beta + \frac{a^2}{r^3} \left[\left\{ 2 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z^2}{r^2} \right. \right. \\ &\quad \left. \left. + \frac{Q^2 - L^2}{4r^2} (x^2 + y^2) \right\} n_\alpha - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z}{r} \delta_\alpha^3 \right] + \frac{a^3}{2r^4} \left[\left(M - \frac{Q^2 - L^2}{r} \right) \right. \\ &\quad \left. - \left(5M - \frac{3(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right] \epsilon_{\alpha\beta 3} n^\beta + \frac{a^4}{8r^5} \left[3 \left\{ \frac{Q^2 - L^2}{2r} + \left(8M - \frac{7(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} - \left(16M \right. \right. \right. \\ &\quad \left. \left. - \frac{21(Q^2 - L^2)}{2r} \right) \frac{z^4}{r^4} \right\} n_\alpha - 2 \left\{ 2 \left(2M - \frac{3(Q^2 - L^2)}{2r} \right) - \left(12M - \frac{7(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \frac{z}{r} \delta_\alpha^3 \right] = -\delta_{l\alpha} b^{(\alpha)}_0, \\ e^{(l)}_\alpha &= \delta^l_\alpha + \left(\frac{n^l n_\alpha}{r} - \frac{2a}{r^2} \epsilon^{(l)}_{\beta 3} n_\alpha n^\beta \right) \left(M - \frac{Q^2 - L^2}{2r} \right) - \frac{a^2}{r^3} \left[\frac{1}{2} \left\{ M + \left(5M - \frac{3(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} n^l n_\alpha \right. \\ &\quad \left. - \left(M - \frac{Q^2 - L^2}{2r} \right) \epsilon^l_{\beta 3} \epsilon_{\alpha\gamma}^3 n^\beta n^\gamma - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{2z}{r} n^{(l)} \delta_\alpha^3 \right] + \frac{a^3}{r^4} \left[\left\{ 6 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z^2}{r^2} \right. \right. \\ &\quad \left. \left. + \frac{Q^2 - L^2}{2r^3} (x^2 + y^2) \right\} n^\beta \epsilon^{(l)}_{\beta 3} n_\alpha - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{2z}{r} n^\beta \epsilon^{(l)}_{\beta 3} \delta_\alpha^3 \right] - \frac{a^4}{8r^5} \left[\left\{ \left(5M - \frac{4(Q^2 - L^2)}{r} \right) \right. \right. \\ &\quad \left. \left. - \left(14M \frac{z^2}{r^2} + \left(63M - \frac{40(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right) \right\} n^l n_\alpha - 8 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) \right. \right. \\ &\quad \left. \left. - \left(7M - \frac{4(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \frac{2z}{r} n^{(l)} \delta_\alpha^3 - 4 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(8M - \frac{5(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right. \right. \\ &\quad \left. \left. + \left(7M - \frac{4(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right\} \delta^l_\alpha + 4 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(9M - \frac{5(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \delta^{3l} \delta_{3\alpha} \right], \quad (29) \end{aligned}$$

and the contravariant components of the parallel vector fields (22) have the form

$$\begin{aligned} e_{(0)}^0 &= 1 + \frac{1}{r} \left(M - \frac{Q^2 - L^2}{2r} \right) + \frac{a^2}{2r^3} \left[\left(M - \frac{Q^2 - L^2}{r} \right) - \left(3M - \frac{2(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right] \\ &\quad + \frac{a^4}{8r^5} \left[\left(3M - \frac{4(Q^2 - L^2)}{r} \right) - \left(30M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} + \left(35M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right], \end{aligned}$$

$$\begin{aligned}
e_{(0)}^\alpha &= \delta^{\alpha l} b^{(0)}_\alpha = -b_{(\alpha)}^0, \\
e_{(l)}^\alpha &= \delta_l^\alpha - \left(\frac{n_l n^\alpha}{r} - \frac{2a}{r^2} \epsilon_{\beta 3(l} n^\alpha) n^\beta \right) \left(M - \frac{Q^2 - L^2}{2r} \right) + \frac{a^2}{r^3} \left[\frac{1}{2} \left\{ M + \left(5M - \frac{3(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} n_l n^\alpha \right. \\
&\quad - \left(M - \frac{Q^2 - L^2}{2r} \right) \epsilon_{l\beta 3} \epsilon_\gamma{}^{3\alpha} n^\beta n^\gamma - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{2z}{r} n_{(l} \delta^{\alpha)3} \left. \right] - \frac{a^3}{r^4} \left[\left\{ 6 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z^2}{r^2} \right. \right. \\
&\quad + \left. \left. \frac{Q^2 - L^2}{2r^3} (x^2 + y^2) \right\} n^\beta \epsilon_{\beta 3(l} n^\alpha) - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{2z}{r} n^\beta \epsilon_{\beta 3(l} \delta^{\alpha)3} \right] + \frac{a^4}{8r^5} \left[\left\{ \left(5M - \frac{4(Q^2 - L^2)}{r} \right) \right. \right. \\
&\quad - \left(14M \frac{z^2}{r^2} + \left(63M - \frac{40(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right\} n_l n^\alpha - 8 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(7M - \frac{4(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \\
&\quad \frac{2z}{r} n_{(l} \delta^{\alpha)3} - 4 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(8M - \frac{5(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} + \left(7M - \frac{4(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right\} \delta_l^\alpha \\
&\quad \left. + 4 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(9M - \frac{5(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \delta_{3l} \delta^{3\alpha} \right]. \tag{30}
\end{aligned}$$

Also with this approximation the covariant components of the metric tensor have the form

$$\begin{aligned}
g_{00} &= 1 - \frac{2}{r} \left(M - \frac{Q^2 - L^2}{2r} \right) - \frac{a^2}{r^3} \left[\left(M - \frac{Q^2 - L^2}{r} \right) - \left(3M - \frac{2(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right] \\
&\quad - \frac{a^4}{4r^5} \left[\left(3M - \frac{4(Q^2 - L^2)}{r} \right) - \left(30M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} + \left(35M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right], \\
g_{0\alpha} &= - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{2n_\alpha}{r} + \frac{2a}{r^2} \left(M - \frac{Q^2 - L^2}{2r} \right) \epsilon_{\alpha\beta 3} n^\beta + \frac{2a^2}{r^3} \left[\left\{ 2 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z^2}{r^2} + \frac{Q^2 - L^2}{4r^2} \right. \right. \\
&\quad \left. \left. (x^2 + y^2) \right\} n_\alpha - \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z}{r} \delta_\alpha{}^3 \right] + \frac{a^3}{r^4} \left[\left(M - \frac{Q^2 - L^2}{r} \right) - \left(5M - \frac{3(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right] \epsilon_{\alpha\beta 3} n^\beta \\
&\quad + \frac{a^4}{4r^5} \left[3 \left\{ \frac{Q^2 - L^2}{2r} + \left(8M - \frac{7(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} - \left(16M - \frac{21(Q^2 - L^2)}{2r} \right) \frac{z^4}{r^4} \right\} n_\alpha \right. \\
&\quad \left. - 2 \left\{ 2 \left(2M - \frac{3(Q^2 - L^2)}{2r} \right) - \left(12M - \frac{7(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \frac{z}{r} \delta_\alpha{}^3 \right], \\
g_{\alpha\beta} &= -\delta_{\alpha\beta} - 2 \left(\frac{n_\alpha n_\beta}{r} - \frac{2a}{r^2} \epsilon_{\gamma 3(\beta} n_\alpha) n^\gamma \right) \left(M - \frac{Q^2 - L^2}{2r} \right) + \frac{a^2}{r^3} \left[\left\{ M + \left(5M - \frac{3(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} n_\beta n_\alpha \right. \\
&\quad + 2 \left(M - \frac{Q^2 - L^2}{2r} \right) \epsilon_{\epsilon\beta 3} \epsilon_{\alpha\gamma}{}^3 n^\epsilon n^\gamma - 4 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z}{r} n_{(\beta} \delta_{\alpha)}{}^3 \left. \right] - \frac{a^3}{r^4} \left[\left\{ 12 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z^2}{r^2} \right. \right. \\
&\quad + \left. \left. \frac{Q^2 - L^2}{r^3} (x^2 + y^2) \right\} n^\gamma n_{(\alpha} - 4 \left(M - \frac{Q^2 - L^2}{2r} \right) \frac{z}{r} n^\gamma \delta_{(\alpha)}{}^3 \right] \epsilon_{\beta)\gamma 3} + \frac{a^4}{4r^5} \left[\left\{ \left(5M - \frac{4(Q^2 - L^2)}{r} \right) \right. \right. \\
&\quad - \left(14M \frac{z^2}{r^2} + \left(63M - \frac{40(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right\} n_\beta n_\alpha - 16 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(7M - \frac{4(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \\
&\quad \frac{z}{r} n_{(\beta} \delta_{\alpha)}{}^3 - 4 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(8M - \frac{5(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} + \left(7M - \frac{4(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} \right\} \delta_{\beta\alpha} \\
&\quad \left. + 4 \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) - \left(9M - \frac{5(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right\} \delta_{\beta 3} \delta_{3\alpha} \right], \tag{31}
\end{aligned}$$

and the asymptotic form of the electromagnetic potential A_μ has the form

$$A_0 = -\frac{q}{4\pi r} + O\left(\frac{1}{r^3}\right), \quad A_\alpha = -\frac{q}{4\pi} \frac{x^\alpha}{r^2} + O_\alpha\left(\frac{1}{r^3}\right). \quad (32)$$

The determinant of the metric tensor is

$$g = -1. \quad (33)$$

Now we evaluate all the required components of the contorsion and the basic vector using ((2) and (4)) neglecting terms beyond the fourth powers of a .

Substituting Eq. (31) and all components of the contorsion and basic vector in (24) we get

$$\begin{aligned} \mathcal{U}_0^{0\alpha} = & \frac{2n^\alpha}{\kappa r^2} \left[\left(M - \frac{Q^2 - L^2}{2r} \right) + \frac{a^2}{r^2} \left\{ M \left(1 - \frac{5z^2}{r^2} \right) - \frac{Q^2 - L^2}{r} \left(1 - \frac{3z^2}{r^2} \right) \right\} + \frac{a^4}{8r^4} \left(M \left\{ \left(9 - 126 \frac{z^2}{r^2} + \frac{189z^4}{r^4} \right) \right\} \right. \right. \\ & \left. \left. - \frac{Q^2 - L^2}{r} \left\{ \left(12 - \frac{96z^2}{r^2} + \frac{120z^4}{r^4} \right) \right\} \right) \right] + \frac{4z\delta^\alpha_3}{\kappa r^3} \left[\frac{a^2}{r^2} \left(M - \frac{Q^2 - L^2}{2r} \right) + \frac{a^4}{4r^4} \left(M \left(9 - \frac{21z^2}{r^2} \right) \right. \right. \\ & \left. \left. - 6 \frac{Q^2 - L^2}{r} \left(1 - \frac{2z^2}{r^2} \right) \right) \right] - \frac{Mn_\beta \epsilon^{\alpha\beta 3}}{\kappa r^2} \left(\frac{a}{r} + \frac{a^3}{r^3} \left(1 - 3 \frac{z^2}{r^2} \right) \right), \end{aligned} \quad (34)$$

and

$$\begin{aligned} \mathcal{U}_\alpha^{0\beta} = & \frac{1}{2\kappa r^2} \left[\frac{Q^2 - L^2}{r} \delta_\alpha^\beta + 2 \left(2M - \frac{3(Q^2 - L^2)}{2r} \right) n_\alpha n^\beta - \frac{a}{r} \left\{ \left(2M - \frac{Q^2 - L^2}{r} \right) \epsilon_{\alpha 3}^\beta + 2 \left(3M \right. \right. \right. \\ & \left. \left. - \frac{2(Q^2 - L^2)}{r} \right) \epsilon_{\alpha \gamma 3} n^\gamma n^\beta \right\} + \frac{a^2}{r^2} \left\{ 2 \left(M - \frac{Q^2 - L^2}{r} \right) \left(\delta_\alpha^\beta - \left(1 - 3 \frac{Q^2 - L^2}{2r} \right) \delta_\alpha^3 \delta_3^\beta \right) - \left(24M \right. \right. \\ & \left. \left. - \frac{35(Q^2 - L^2)}{2r} \right) \frac{z^2}{r^2} n_\alpha n^\beta + \frac{5(Q^2 - L^2)}{2r} \epsilon_{\alpha \gamma 3} \epsilon_\delta^{3\beta} n^\gamma n^\delta + 2 \left(8M - \frac{15(Q^2 - L^2)}{2r} \right) \frac{z}{r} n^{(\beta} \delta_{\alpha)}^3 \right\} \\ & - \frac{a^3}{r^3} \left\{ \left(M - \frac{Q^2 - L^2}{r} \right) \epsilon_{\alpha 3}^\beta + \left[\left(5M - \frac{6(Q^2 - L^2)}{r} \right) - \left(35M - \frac{24(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \right] \left(\epsilon_{\alpha \gamma 3} n^\gamma n^\beta \right. \right. \\ & \left. \left. + \left(10M - \frac{6(Q^2 - L^2)}{r} \right) \frac{z}{r} n^\rho \epsilon_{\alpha \rho} \delta_3^\beta \right) - \left(5M - \frac{3(Q^2 - L^2)}{r} \right) \frac{z^2}{r^2} \epsilon_{\alpha 3}^\beta \right\} + \frac{a^4}{r^4} \left\{ \left[\left(2M - \frac{9(Q^2 - L^2)}{4r} \right) \right. \right. \\ & \left. \left. - \left(54M - \frac{357(Q^2 - L^2)}{8r} \right) \frac{z^2}{r^2} + \left(168M - \frac{126(Q^2 - L^2)}{r} \right) \frac{z^4}{r^4} - \left(120M - \frac{693(Q^2 - L^2)}{8r} \right) \frac{z^6}{r^6} \right] \right. \\ & \left(\delta_\alpha^\beta - \left(1 - 15 \frac{Q^2 - L^2}{8r} \right) \delta_\alpha^3 \delta_3^\beta \right) + \left[\frac{21(Q^2 - L^2)}{8r} + \left(48M - \frac{189(Q^2 - L^2)}{4r} \right) \frac{z^2}{r^2} - \left(120M \right. \right. \\ & \left. \left. - \frac{693(Q^2 - L^2)}{8r} \right) \frac{z^4}{r^4} \right] \epsilon_{\alpha \gamma 3} \epsilon_\delta^{3\beta} n^\gamma n^\delta + 2 \left[\left(12M - \frac{105(Q^2 - L^2)}{8r} \right) - \left(96M - \frac{315(Q^2 - L^2)}{4r} \right) \frac{z^2}{r^2} \right. \\ & \left. \left. + \left(120M - \frac{693(Q^2 - L^2)}{8r} \right) \frac{z^4}{r^4} \right] \frac{z}{r} n^{(\beta} \delta_{\alpha)}^3 + \left[\left(12M - \frac{105(Q^2 - L^2)}{8r} \right) \frac{z^2}{r^2} \right. \right. \\ & \left. \left. + \left(48M - \frac{315(Q^2 - L^2)}{4r} \right) \frac{z^4}{r^4} - \left(120M - \frac{693(Q^2 - L^2)}{8r} \right) \frac{z^6}{r^6} \right] \delta_{\alpha 3} \delta_3^\beta \right\} \right]. \end{aligned} \quad (35)$$

The components we are interested in of the energy-momentum density are given by

$$\begin{aligned}\tau_0^0 &= \frac{Q^2 - L^2}{\kappa r^4} \left[1 - \frac{a^2}{r^2} \left(2 - \frac{6}{r^2} (x^2 + y^2) \right) + \frac{a^4}{r^4} \left(3 - \frac{24}{r^2} (x^2 + y^2) + \frac{30}{r^4} (x^2 + y^2)^2 \right) \right], \\ \tau_\alpha^0 &= -\frac{2a(Q^2 - L^2)\epsilon_{\alpha\beta 3}n^\beta}{\kappa r^5} \left(1 + \frac{3a^2}{r^2} (r^2 - 2z^2) \right).\end{aligned}\quad (36)$$

Further substituting (36) in (28) and then transforming it into spherical coordinate we obtain

$$E = \frac{(Q^2 - L^2)}{\kappa r^6} \int_{r=R} \int \int \left(r^4 + 4a^2 r^2 - 6a^2 r^2 \cos^2 \theta + 9a^4 - 36a^4 \cos^2 \theta + 30a^4 \cos^4 \theta \right) dr \sin \theta d\theta d\phi. \quad (37)$$

Performing the above integration we get the energy associated with the exterior general axisymmetric black hole given by Eq. (22) in the form

$$E(R)_{total}^{exterior} = \frac{(Q^2 - L^2)}{R} \left(\frac{1}{2} + \frac{a^2}{3R^2} + \frac{3a^4}{10R^4} \right) + O\left(\frac{1}{R^6}\right). \quad (38)$$

Eq. (38) shows that the total energy associated with the spacetime given by Eq. (22) is shared by its exterior as well as interior.

Now we turn our attention to the angular momentum of the solution given by Eq. (22). From (36) we can get the components of the momentum density in the form

$$P_\alpha = -\frac{2a(Q^2 - L^2)\epsilon_{\alpha\beta 3}n^\beta}{\kappa r^5} \left(1 + \frac{3a^2}{r^2} (r^2 - 2z^2) \right), \quad (39)$$

from which we can show that there is no momentum-density associated with Kerr black hole i.e., when $Q = L = 0$. The components of the angular momentum of a general-relativistic system is given by [40]

$$J_\alpha = \int \int \int (x_\beta P_\gamma - x_\gamma P_\beta) d^3x, \quad (40)$$

where α, β, γ take cyclic values 1,2,3. Using (39) in (40) and transforming the expressions into spherical coordinates

$$J_\alpha = \int \int \int \frac{2a(Q^2 - L^2)}{\kappa r^4} \left(r^2 + 3a^2(1 - 2\cos^2 \theta) \right) \sin^3 \theta dr d\theta d\phi, \quad (41)$$

performing the above integration between two spheres of radii R_1 and R_2 we get

$$J_\alpha = -2a(Q^2 - L^2)\epsilon_{12\alpha} \left[\frac{1}{3r} + \frac{a^2}{5r^3} \right]_{R_1}^{R_2}. \quad (42)$$

Eq. (42) shows that the rotation of the charged object is responsible for the angular momentum distribution due to the electromagnetic field and the NUT parameter.

5. Main results and discussion

In this paper we have studied the coupled equations of the gravitational and electromagnetic fields in the tetrad theory of gravitation. Applying the most general tetrad (15) to the field equations (7)~(9) we have obtained an exact solution given by Eq. (22). Eq. (22) is a general axi-symmetric solution from which we can generate the other known solutions like Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman spacetimes. The metric tensor associated with Eq. (22) is the Kerr-NUT spacetime.

It was shown by Møller [41] that the tetrad description of the gravitational field allows a more satisfactory treatment of the energy-momentum complex than does GR. Therefore, we have used the superpotential (24), *formulated within the framework of teleparallel spacetime* to calculate the energy-momentum density given by Eq. (27). We have used this energy-momentum density to evaluate the total exterior energy of the gravitating charged rotating body given by Eq. (22). Because the definition of energy of Eq. (28) requires its evaluation in Cartesian coordinate, the calculations without any approximation is obviously very tedious. Moreover, the intrinsic rotation parameter a is quantitatively very small for most physical situations. Therefore, for our convenience we keep terms containing powers of a up to the fourth order. With this approximation we have calculated the components of a covariant and contravariant tetrad fields (29) and (30), a covariant components of metric tensor (31). Calculating all the necessary components of the contorsion and basic vector ((2) and (4)) and using (31) in (28) we have obtained the expression of the exterior energy of the general black hole given by Eq. (22) till the fourth order. When the rotation, the charge as well as the NUT parameters are considered we get an additional terms

$$\frac{Q^2 - L^2}{R} \left(\frac{1}{2} + \frac{a^2}{3R^2} + \frac{3a^4}{10R^4} \right), \quad (43)$$

which is the energy of the exterior magnetic field due to the rotation of the charged object. The asymptotic value of the total gravitational mass of a Kerr-NUT spacetime is the ADM [42] therefore, the energy associated with a Kerr-NUT contained in a sphere of radius R is

$$E(R) = M - \frac{Q^2 - L^2}{R} \left(\frac{1}{2} + \frac{a^2}{3R^2} + \frac{3a^4}{10R^4} \right). \quad (44)$$

Switching off the rotation and NUT parameters we found that the energy given by Eq. (44) will be the same as of Reissner-Nordström metric [43, 44]. The energy of Eq. (44) is confined to its interior only when we set the charge and the NUT parameters to be zero. This result is quit in conformity with those of Virbhadra [45, 46] and Cooperstock et al. [47] and Ahmed [48].

Using the value of energy-momentum density given by Eq. (36) we have calculated the momentum density. Also we have calculated the angular momentum distribution due to the electromagnetic field present in the Kerr-NUT field using Eq. (39) in Eq. (40). It is clear from Eq. (42) that the momentum density depends mainly on the rotation of the charged object and has only one component. As is clear from Eq. (42) that the angular

momentum depends on the even powers of the charge and NUT parameters and odd powers of the rotation parameter which means that the direction of the angular momentum vector depends on the direction of the rotation parameter.

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